Tessellation

The purity of geometry

Photograph by Francisco Anzola, CC BY-SA

Why is the Sydney Opera House tiled?

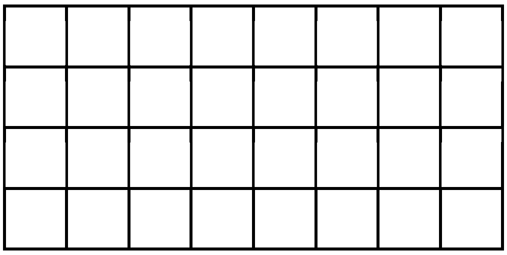
When Jorn Utzon designed the beautifully curved sails of the Sydney Opera House, he carefully planned the geometry and materials of the sail surfaces to respond dynamically to the sunlight of Sydney. Unlike the soft and muted shades of light in his native Denmark, Utzon recognised that this part of Australia is characterised by strong and harsh light, especially in the peak of summer.

One of Utzon's key insights was that if he were to give the Sydney Opera House sails a perfectly smooth surface, either of bare concrete or painted in some solid colour, the spherical shape of the sails would be invisible when cast in harsh light. He wanted sunlight to reflect at different angles from every point on the sails, no matter what time of day it was or where the sun was shining from. This was a major factor in his choice to cover the Sydney Opera House in tiles.

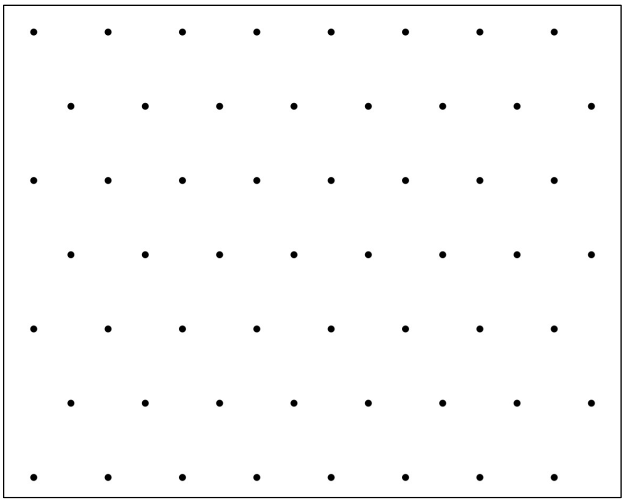
The mathematics of covering a flat surface (or "plane") in geometric patterns is called tessellation. It gets its name from the Latin word *tessera*, which literally means "four-sided stone block" - in other words, a tile!

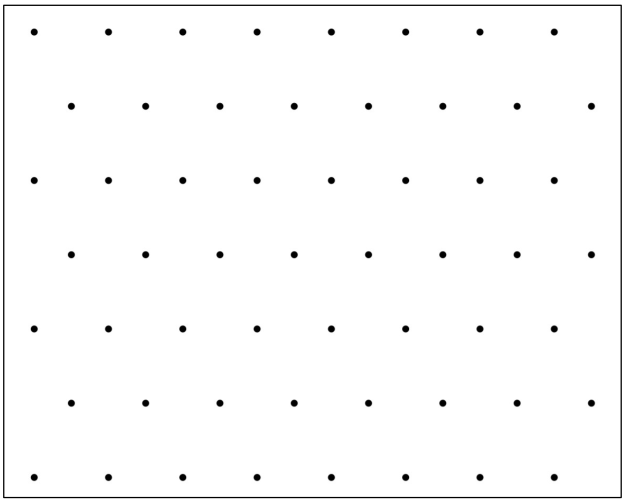
Regular tessellations

The simplest kind of tessellation is called a *regular tessellation*, in which we cover the plane entirely with copies of a single shape that has equal sides. For instance, we can create a regular tessellation using squares:



There are only two other kinds of regular tessellation possible: one using *equilateral triangles* and one using *regular hexagons*. Use the isometric grids below to help you create these regular tessellations!





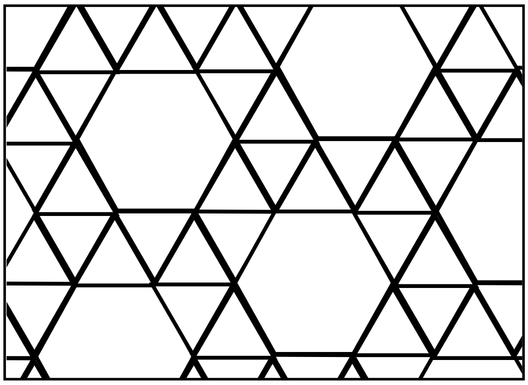
While there are an infinite number of other regular polygons (shapes with straight sides that are all equal in length), only the ones above (squares, triangles and hexagons) can form a regular tessellation.

It's impossible to tessellate with regular pentagons (5 sides), heptagons (7 sides), octagons (8 sides) or any other number. Try to form a tessellation with one of these in the blank space below and see if you can determine why it can't be done!

Hint: if you're stuck, have a think about the angles in a regular polygon. Remember that the sum of all angles at a vertex (where the sides of different shapes meet) must add up to exactly 360°. You can find more information about this later on in this worksheet!

Semi-regular tessellations

If you form a pattern with more than one kind of regular polygon, this is called a *semi-regular tessellation*. You can use as many kinds of regular polygon as you like, but the pattern at each vertex must be the same. Here's an example:



This semi-regular tessellation is made up of triangles and hexagons. There is another semi-regular tessellation that is also entirely made up of triangles and hexagons, but it looks quite different. To tell them apart, tessellations are named in the following way:

1. Select a vertex somewhere in the pattern. (Remember that since every vertex is the same, it doesn't matter which one you select!)
2. Pick one of the polygons that connects to that vertex and write down the number of sides it has (e.g. "3"). Then move clockwise and continue writing down the number of each successive polygon until you come back to the start.
3. To communicate consistently, mathematicians usually start counting at the polygon with the least number of sides – so the pattern above would be called "3-3-3-3-6".

The other semi-regular tessellation that is composed of triangles and hexagons is called "3-6-3-6". Can you use this fact to draw it in the blank space below?

Including the two shown above, there are only eight semi-regular tessellations that can be formed. Can you use the following clues to work out the remaining six?

- Two of the tessellations only use triangles and squares
- One uses triangles and dodecagons (those are polygons with 12 sides!)
- One uses squares and octagons
- One uses triangles, squares and hexagons
- One uses squares, hexagons and dodecagons

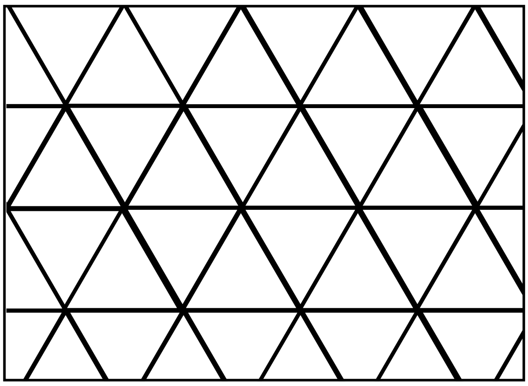
Angles of regular polygons

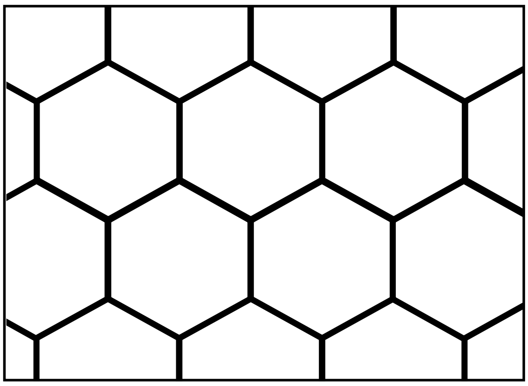
If you're having trouble coming up with these simply by drawing shapes, you might find it useful to think about this problem in a more systematic way by thinking about what the angles in a regular polygon are equal to. As we increase the number of sides in a polygon, we also increase the size of each angle inside the polygon. Fill in the blanks of the table below to help you calculate the size of the angles in each shape.

Regular polygon	How many sides in the shape?	What is the sum of all the angles in the shape?	What is the size of each individual angle in the shape?
Equilateral triangle	3	180°	180° ÷ 3 = 60°
Square		360°	90°
Regular pentagon	5	540°	
Regular hexagon	6		
Regular heptagon	7	900°	
Regular octagon		1080°	
Regular nonagon	9		140°
Regular decagon		1440°	144°
Regular undecagon	11		
Regular dodecagon		1800°	150°

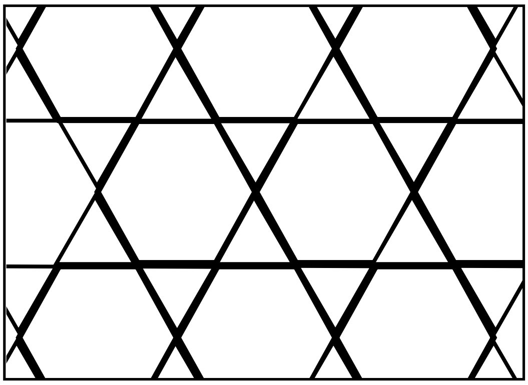
Solutions

Here are the regular tessellations for triangles and hexagons:





Here is the second semi-regular tessellation made up of triangles and hexagons:



And here are the remaining six semi-regular tessellations: