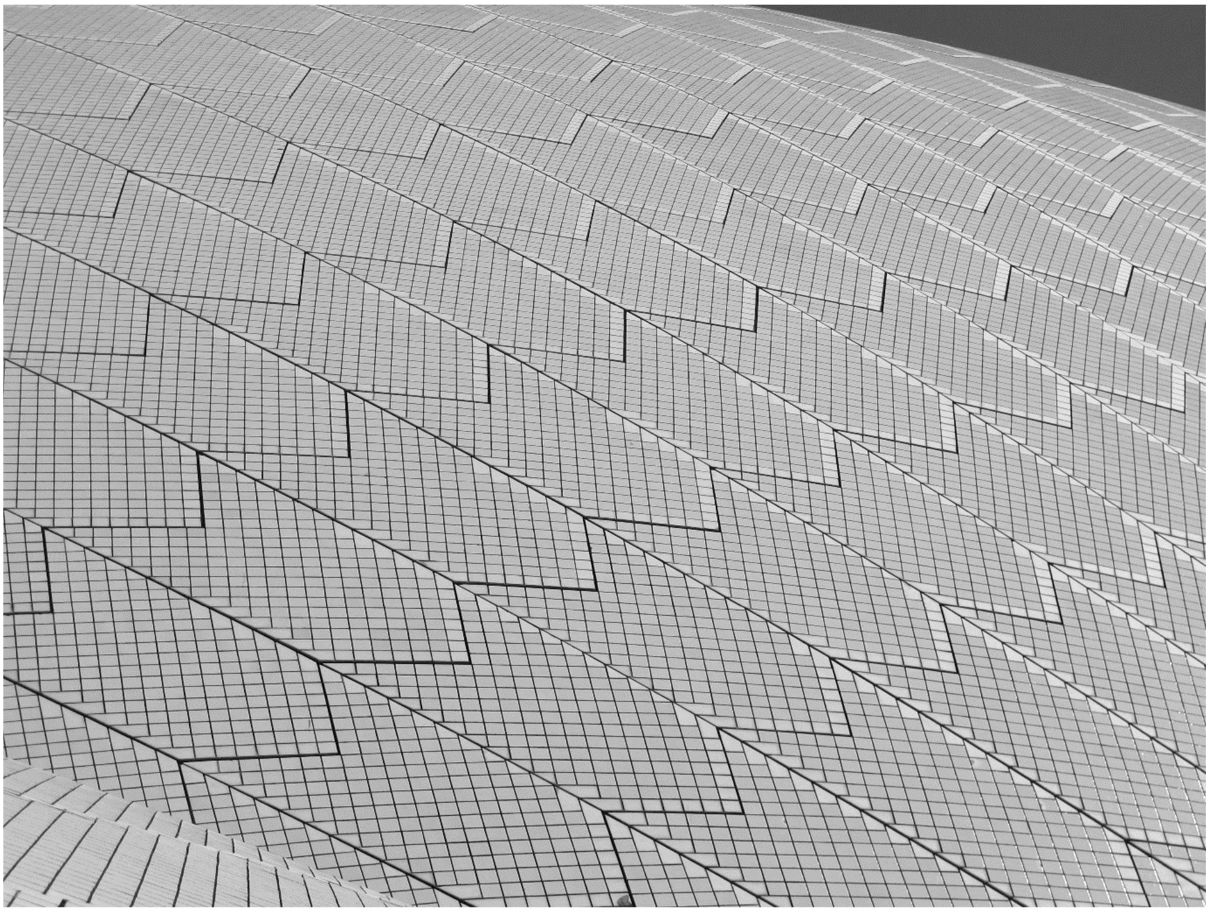
Roof Tile Equations

Calculating an impossible number



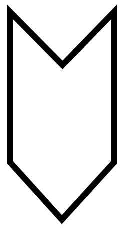
Photograph by Albert Straub, CC BY-SA

How many tiles are there?

From a distance, the Sydney Opera House appears to be a relatively simple white building. As we get closer, though, the detailed textures of the iconic sails become clearer. Light reflects in many beautiful ways from the surfaces of the Sydney Opera House because its spherical shape is accentuated by two features that work together:

- A distinctive chevron pattern
- The combination of glossy tiles and matte tiles that reflect the sun's light in different ways depending on its angle

A chevron is a hexagonal figure that looks like the head of an arrow (one is shown to the right here). There are 4228 of these chevrons that cover the sails of the Sydney Opera House, and they are made in 26 different sizes. This is because chevrons closer to the ground are narrow, while chevrons near the top of the sails are much wider. The chevrons increase in width as we climb the structure.



Most of the tiles are glossy, and they give off a very bright reflection of the sun when it shines on the Sydney Opera House at just the right angle. However, the lower and lateral (side) edges of the chevrons are tiled with materials that are less reflective. These matte tiles follow a simple geometric pattern that allows us to calculate how many matte tiles are needed using some algebraic equations.

Lower edge tiles

The lower edge of each chevron, except the first chevron (which is closest to the ground), always has an odd number of square matte tiles. If we were to list them out, they would form the following sequence:

$$1, 3, 5, 7, 9 \dots$$

While this is easy to do for a few numbers, as the list becomes longer it becomes more difficult to keep track of all the numbers – especially the larger ones. This is why we use algebra to express the pattern of odd numbers:

Chevron number	**How many matte tiles on this chevron's lower edge?**
1	Doesn't have any – so it doesn't fit into this pattern!
2	1
3	3
4	5
5	7
…	
n	$2n - 3$

This kind of number sequence, where there is a *common difference* between each successive term of the sequence, is called an *arithmetic progression.* The most common type of arithmetic progression that we all learn from a young age are our times tables!

Try substituting various values for n and see how they lead to the correct number of matte tiles that we can observe on the Sydney Opera House chevrons. Go back to the photo on the first page of this activity and see if you can determine which numbered chevrons you are looking at, by comparing it with the number of matte tiles on its lower edge. You'll need to *change the subject* of your equation, like this:

e.g. If $L_n = 2n - 3$ and a particular chevron has 17 tiles on its lower edge, then:

$2n - 3 = 17$

$2n = 20$ (adding 3 to both sides)

$n = 10$ (dividing both sides by 2)

The equation $L_n = 2n - 3$ allows us to determine the number of matte tiles on the lower edge of any specific chevron we choose. However, if we wanted to determine the total number of matte tiles on the lower edges of several chevrons – as Utzon and his team needed to do so that they could order the correct number of tiles to be manufactured in Sweden – we would use a different formula:

$$\text{Total} = \frac{n}{2}[1 + (2n - 3)]$$

This equation is formed by pairing up the first term (1) with the last term $(2n - 3)$ in the sequence, the second term with the second-last term in the sequence, and so on. Each of these pairings would be equal in value and since there are n terms in the entire sequence, there would be $\frac{n}{2}$ pairs. Evaluating this would give us the total number of matte tiles on the lower edges for the chevrons up to any height we specify.

Try expanding the algebraic expression on the right-hand-side of the equation above to simplify the equation and make it easier to use!

Lateral edge tiles

There are matte tiles made of the same materials on the side edges of each chevron. Though they are a different shape (not square like the other tiles), their quantity can still be calculated using the help of algebra.

Look back again at the photograph on the first page of this activity. Count carefully and see if you can work out the number of matte tiles on the *side* of each chevron. While it may initially appear that the chevrons become shorter as you go further up the structure, this is actually an optical illusion due to the chevrons becoming wider the higher you go. In reality, you should find that some chevrons are 12 tiles high while others are 13 tiles high. Since each chevron has an equal left edge and an equal right edge, this means that there will be either 24 or 26 tiles on the sides of each chevron.

To model these numbers in an equation, we need to use an algebraic tool called a *switching factor*. This combines two concepts that you might already be familiar with: *indices* and *negative numbers*.

The most basic definition of indices is to think of them as repeated multiplication. Recall that $2^5 = 2 \times 2 \times 2 \times 2 \times 2$, which equals 32. If we introduce negative numbers into our indices, we need to take care as several negative signs will often cancel out. For instance, $(-2)^5 = (-2) \times (-2) \times (-2) \times (-2) \times (-2)$, which equals -32.

This is the concept that drives the equation for the lateral edge tiles:

$$S_n = 25 - (-1)^n$$

The switching factor is the final term on the right-hand-side of the equation, $(-1)^n$, which *switches values* between 1 (for even values of n) and -1 (for odd values of n). Here's an example of how it works, looking at chevron 4:

If $n = 4$, then $S_4 = 25 - (-1)^4$

$$= 25 - [(-1) \times (-1) \times (-1) \times (-1)]$$

$$= 25 - 1$$

$$= 24$$

Can you use a similar method to prove that all odd-numbered chevrons will have 26 matte tiles on their side edges?